Stepeni (potencijalni) redovi

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots \quad \text{ili} \qquad (1)$$

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1 (x-x_0) + a_2 \cdot (x-x_0)^2 + \dots \qquad (2)$$

$x - x_0 = y \Rightarrow$ red (2) dobija oblik reda (1)

Th 1 Za svaki red (1) postoji broj $R \geq 0$ tako da red konvergira unutar intervala $(-R, R)$ a divergira van tog intervala.

R - poluprečnik (radijus) konvergencije

$(-R, R)$ - oblast (interval, krug) konvergencije

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \quad \text{ili} \quad R = \lim_{n\to\infty} \sqrt[n]{|a_n|}^{\,-1} = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}}$$

① Naći poluprečnik i interval konvergencije reda:

a) $\displaystyle\sum_{n=1}^{\infty} \underbrace{\left(\frac{2n-1}{3n+2}\right)^n}_{a_n} \cdot (x+2)^n$

$$R = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{\left(\frac{2n-1}{3n+2}\right)^n}} = \lim_{n\to\infty} \frac{1}{\frac{2n-1}{3n+2}} = \lim_{n\to\infty} \frac{3n+2}{2n-1} = \frac{3}{2}$$

$x \in (-R, R)$
$|x| < R$

$$-\frac{3}{2} < x+2 < \frac{3}{2}$$

$$-2-\frac{3}{2} < x < \frac{3}{2}-2 \Rightarrow -\frac{7}{2} < x < -\frac{1}{2}$$

$$\Rightarrow x \in \left(-\frac{7}{2}, -\frac{1}{2}\right)$$

$$x = -\frac{7}{2} \Rightarrow \sum_{n=1}^{\infty} \left(\frac{2n-1}{3n+2}\right)^n \cdot \left(-\frac{7}{2}+2\right)^n = \sum_{n=1}^{\infty} \left(\frac{2n-1}{3n+2}\right)^n \cdot \left(-\frac{3}{2}\right)^n$$

$$= \sum_{n=1}^{\infty} (-1)^n \cdot \left(\frac{6n-3}{6n+4}\right)^n$$

$$\lim_{n\to\infty} \left(\frac{6n-3}{6n+4}\right)^n \stackrel{(1)^\infty}{=} \lim_{n\to\infty} \left(\frac{6n+4-7}{6n+4}\right)^n = \lim_{n\to\infty} \left(1 - \frac{7}{6n+4}\right)^n =$$

$$= \lim_{n\to\infty} \left[\left(1 + \frac{1}{\frac{6n+4}{-7}}\right)^{-\frac{6n+4}{7}}\right]^{\frac{n}{-\frac{6n+4}{7}}} = e^{\lim_{n\to\infty} -\frac{7n}{6n+4}} = e^{-\frac{7}{6}} \neq 0$$

red divergira jer opći član ne $\to 0$!

$$x = -\frac{1}{2} \Rightarrow \sum_{n=1}^{\infty} \left(\frac{2n-1}{3n+2}\right)^n \cdot \left(-\frac{1}{2}+2\right)^n = \sum_{n=1}^{\infty} \left(\frac{2n-1}{3n+2}\right)\left(\frac{3}{2}\right)^n =$$

$$= \sum_{n=1}^{\infty} \left(\frac{6n-3}{6n+4}\right)^n \Rightarrow \ldots$$ ponovo, opći član ne $\to 0$, pa red divergira!

* $x \in \left(-\frac{7}{2}, -\frac{1}{2}\right)$ - interval konvergencije

b) $\sum_{n=1}^{\infty} \frac{(3n-2)(x-3)^n}{(n+1)^2 \; 2^{n+1}}$

$$R = \lim_{n\to\infty} \left| \frac{\frac{3n-2}{(n+1)^2 2^{n+1}}}{\frac{3(n+1)-2}{(n+1+1)^2 \; 2^{n+1+1}}} \right| = \lim_{n\to\infty} \left| \frac{\frac{3n-2}{(n+1)^2 2^{n+1}}}{\frac{3n+1}{(n+2)^2 \; 2^{n+2}}} \right| =$$

$$= \lim_{n\to\infty} \left| \frac{(3n-2)(n+2)^2 \; \cancel{2^n} \, 2^{\cancel{2}}}{(n+1)^2 (3n+1) \; \cancel{2^n} \cdot \cancel{2}} \right| = \lim_{n\to\infty} \left| \frac{(3n-2)(n+2)^2 \cdot 2}{(n+1)^2 (3n+1)} \right| =$$

$$= \lim_{n\to\infty} \frac{6n^3}{3n^3} = 2$$

$x \in (-R, R)$

$$-2 < x-3 < 2$$
$$-2+3 < x < 2+3$$
$$1 < x < 5$$

$x \in (1,5)$

$$x = 1: \quad \sum_{n=1}^{\infty} \frac{(3n-2)(1-3)^n}{(n+1)^2 \; 2^{n+1}} = \sum_{n=1}^{\infty} \frac{(3n-2)(-2)^n}{(n+1)^2 \; 2^n \cdot 2} =$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{(3n-2) \cancel{2^n}}{(n+1)^2 \cdot \cancel{2^n} \cdot 2} = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{(3n-2)}{2 \, (n+1)^2}$$

po Leibnitzovom

$\Rightarrow$

$$\frac{3}{2 \cdot (n+1)^2} \sim \frac{3n}{2n^2} = \frac{3}{2n} \Rightarrow \sum_{n=1}^{\infty} (-1)^n \cdot \frac{3}{2n} \text{ - konvergira}$$

$x=5$: $\sum_{n=1}^{\infty} \frac{(3n-2)\cancel{2^n}}{(n+1)^2 \cdot 2^{\cancel{n}+1}} = \sum_{n=1}^{\infty} \frac{3n-2}{2(n+1)^2}$

$\frac{3n-2}{2(n+1)^2} \sim \frac{3}{2n}$ - divergira

$\sum \frac{3}{2n}$ - divergira ($\sum \frac{1}{n}$ - harmonijski red divergira)

$x \in [1, 5)$

c) $\sum_{n=2}^{\infty} \frac{x^{n-1}}{3^n \cdot n \cdot \ln n}$

$$R = \lim_{n\to\infty} \left| \frac{\frac{1}{3^n \cdot n \cdot \ln n}}{\frac{1}{3^{n+1} \cdot (n+1) \cdot \ln(n+1)}} \right| = \lim_{n\to\infty} \left| \frac{\cancel{3^n} \cdot 3 \cdot (n+1) \cdot \ln(n+1)}{\cancel{3^n} \cdot n \cdot \ln n} \right|$$

$$= 3 \lim_{n\to\infty} \underbrace{\frac{n+1}{n}}_{\to 1} \cdot \frac{\ln(n+1)}{\ln n} = 3 \cdot \lim_{n\to\infty} \frac{\ln n(1+\frac{1}{n})}{\ln n} =$$

$$= 3 \cdot \lim_{n\to\infty} \frac{\ln n + \ln(1+\frac{1}{n})}{\ln n} = 3 \cdot \lim_{n\to\infty} \left(1 + \frac{\ln(1+\frac{1}{n})^{\to 0}}{\ln n_{\to \infty}}\right) = 3$$

$\frac{0}{\infty} \to 0$

$|x| < 3 \Rightarrow x \in (-3, 3)$

alternativni red

$$x=-3 : \Rightarrow \sum_{n=2}^{\infty} \frac{(-3)^{n-1}}{3^n \cdot n \ln n} = \sum_{n=2}^{\infty} \frac{(-1)^{n-1} \cdot 3^n \cdot 3^{-1}}{3^n \cdot n \cdot \ln n} = \frac{1}{3} \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n \cdot \ln n}$$

konvergira !

$$b_n = \frac{1}{n \ln n}$$

Po Leibniz: 1° da $\to 0$

2° da je opadajući

1° $\lim_{n\to\infty} b_n = 0$

2° $\{b_n\}$ je opadajući:

$n \cdot \ln n$ je rastući jer je n i $\ln n$ rastuća, pa $\frac{1}{n \ln n}$ je opadajući.

$$x=3 : \sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

divergira

$f(x) = \frac{1}{x \ln x}$ $(x \geq 2)$ neprekidna, pozitivna, opadajuća

Primjenimo Košijev integralni kriterij:

$$\int_2^{+\infty} \frac{1}{x \ln x} \, dx = \left| \ln x = t \right| = \dots = +\infty$$

Zaključak: $x \in [-3, 3)$

$\ln x$ je rastuća f-ja pa je i red

za vježbu:

d) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2n+1}$

f) $\sum_{n=1}^{\infty} \frac{(x+3)^n}{n \cdot 5^n}$

e) $\sum_{n=1}^{\infty} \frac{n^n}{n!} x^n$

g) $\sum_{n=1}^{\infty} \frac{(2n-1)!!}{n!} \cdot (x+2)^n$

(2.) Izračunati sumu datog funkcionalnog reda

pod 1) ovo je 1!

a) $\sum_{n=1}^{\infty} (-1)^{n-1} (2n-1) x^{2n-2}$

Th2 Ako je R poluprečnik konvergencije stepenog reda $\sum_{n=0}^{\infty} a_n x^n$ tj red se može proizvoljan broj puta integrirati i diferencirati član po član na intervalu $(-R,R)$.

Diferenciranje član po član ! proizvoljan red

$$\left(\sum_{n=1}^{\infty} f_n(x)\right)' = \sum_{n=1}^{\infty} f_n'(x)$$

Integriranje član po član $\int_a^b \left(\sum_{n=1}^{\infty} f_n(x)\right) dx = \sum_{n=1}^{\infty} \int_a^b f_n(x) dx$

$$R = \lim_{n\to\infty} \frac{2n-1}{2n+1} = 1$$

$|x| < 1 \Rightarrow x \in (-1,1)$

$x = \pm 1 \Rightarrow$ ne konvergira

Označimo sa $S(x) = \sum_{n=1}^{\infty} (-1)^{n-1} (2n-1) x^{2n-2}$

$$S'(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot (2n-1)(2n-2) x^{2n-3}$$ – nije – pogodan za sumiranje !!!

$$\int S(x)dx = \int \sum_{n=1}^{\infty} (-1)^{n-1} (2n-1) x^{2n-2} \, dx =$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \cdot (2n-1) \int x^{2n-2} dx = \sum_{n=1}^{\infty} (-1)^{n-1} \cancel{(2n-1)} \cdot \frac{x^{2n-1}}{\cancel{2n-1}} + C$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \cdot x^{2n-1} + C = x - x^3 + x^5 - x^7 \dots + C$$

geometrijski red sa količnikom $q = -x^2$!

$$= \frac{x}{1-(-x)^2} + C = \frac{x}{1+x^2} + C$$

$$\boxed{S(x) = \left(\frac{x}{1+x^2}\right)' = \frac{1-x^2}{(1+x^2)^2}, \quad x \in (-1,1)}$$

$a = \text{const.} \quad a > 0$

b) $S(x) = \sum_{n=2}^{\infty} \frac{x^n}{n \cdot a^{n-1}}$

$R = a \Rightarrow x \in (-a, a)$

$$S'(x) = \sum_{n=2}^{\infty} \frac{n x^{n-1}}{n a^{n-1}} = \sum_{n=2}^{\infty} \frac{x^{n-1}}{a^{n-1}} = \sum_{n=2}^{\infty} \left(\frac{x}{a}\right)^{n-1} = \frac{x}{a} + \left(\frac{x}{a}\right)^2 + \left(\frac{x}{a}\right)^3 + \dots =$$

geometrijski red $\quad a_n = q = \frac{x}{a} \Big| = \frac{\frac{x}{a}}{1-\frac{x}{a}} = \frac{\frac{x}{a}}{\frac{a-x}{a}} = \frac{x}{a-x}$

$S'(x) = \frac{x}{a-x} \qquad x \in (-a, a)$

$S(x) = \int \frac{x}{a-x}\,dx = \int \left(\frac{x-a}{a-x} + \frac{a}{a-x}\right) dx = -\int dx + a \int \frac{dx}{a-x} =$

$= -x - a \cdot \ln|a-x| + C$

$S(0) = \sum_{n=2}^{\infty} \frac{0}{n \cdot a^{n-1}} = 0$

$S(0) = \sum_{n=2}^{\infty} -x - a \cdot \ln a + C =$

$\Rightarrow -a \ln a + C = 0 \quad \Rightarrow \quad \boxed{C = a \ln a}$

$x < a$
$0 < a - x$ ne treba | |

$S_n(x) = -x - a \cdot \ln(a-x) + a \ln a = -x + a \ln \frac{a}{a-x}$

$S_n(x) = -x + a \cdot \ln \frac{a}{a-x} \quad , \quad x \in (-a, a)$

za vježbu:

c) $\sum_{n=1}^{\infty} \frac{x^{4n-3}}{4n-3}$

e) $\sum_{n=1}^{\infty} n x^n = S(x)$

d) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{n+1}}{n \cdot (n+1)}$

Uputa: $S(x) = x \sum_{n=1}^{\infty} n x^{n-1} = S_n(x)$

$\int S_n(x) = \ldots =$ geometrijski red

2-put diferencirat!

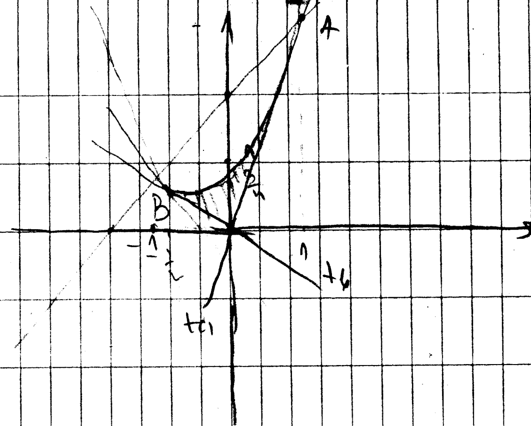
1° iii 2°

2°

f) $\sum_{n=0}^{\infty} \frac{x^n}{n+3}$ $\left(\cdot \sum \frac{x^{n+3}}{n+3} \cdot \frac{1}{x^3} \right) = \frac{1}{x^3} \sum \frac{x^{n+3}}{n+3}$

1° $(x^n)' = n \cdot x^{n-1}$

1. $\begin{cases} y = x^2 + x + 1 \\ y = x + 2 \end{cases}$



$x^2 + x + 1 = (x + \frac{1}{2})^2 + \frac{3}{4} > 0$

$T(-\frac{1}{2}, \frac{3}{4})$

x	-2	0
$x+2$	0	2

$(x^2 + x + 1)' = 2x + 1$

$A(1, 3)$

$B(-1, 1)$

$t_a: \; y - 3 = (\underbrace{2 \cdot 1 + 1}_{k_{ta}})(x - 1)$

$y - 3 = 3x - 3$

$\boxed{y = 3x}$

$t_b: \; y - 1 = (-2 + 1)(x + 1)$

$y - 1 = -x - 1$

$\boxed{y = -x}$

$$P = \int_{-1}^{0} (x^2 + x + 1 - (-x))\, dx + \int_{0}^{1} (x^2 + x + 1 - 3x)\, dx$$

$= \ldots$

2. D.Z. Na parabolu $y = x^2 + 4x + 6$ povučena je tangenta u tački $x = 1$. Izračunati površinu koju obrazuje data parabola sa povučenom tangentom i osom simetrije parabole.

Funkcionalni nizovi

Oznaka: $\{f_n(x)\}$ $\quad f_1(x), f_2(x), f_3(x) \ldots f_n(x)$ - niz funkcija

$f_n(x) \downarrow$ opšti član

Def:

Kažemo da niz $\{f_n(x)\}$ konvergira na skupu S ka funkciji $f(x)$ ako $(\forall \varepsilon > 0)(\forall x \in S)(\exists n_0 = n_0(\varepsilon, x))\; n \geq n_0 \Rightarrow$

$|f_n(x) - f(x)| < \varepsilon$

Tada pišemo $\lim\limits_{n\to\infty} f_n(x) = f(x) \quad (x \in S)$ ili kraće $f_n \xrightarrow[S]{} f$

n_0 - prirodan broj

Ako funkcionalni niz konv. ravnomjerno na nekom skupu, onda on i konvergira na tom skupu a obrnuto ne vrijedi uvijek.

1. Dokazati da dati funkc. niz konvergira na datom skupu i naći limes tog niza

a) $f_n(x) = x^n \qquad S = [0,1]$

$$\lim x^n = \begin{cases} +\infty, & x > 1 \\ 1, & 1 \\ 0, & x \in (-1,1) \end{cases} \qquad f(x) = \begin{cases} 0; & x \in [0,1) \\ 1, & x = 1 \end{cases}$$

b) $f_n(x) = \dfrac{n^2}{n^2 + x^2} \qquad S \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{n^2}{n^2+x^2} = \lim_{n\to\infty} \frac{1}{1+\frac{x^2}{n^2}} = 1$$

$f(x) = 1 \quad (x \in \mathbb{R})$

c) $f_n(x) = n \cdot \sin\frac{1}{nx}$ $\quad S = (0, +\infty)$

$\lim_{n\to\infty} \left(n \cdot \sin\frac{1}{nx}\right) = \infty \cdot 0$

$\lim_{n\to\infty} \left(\frac{\sin\frac{1}{nx}}{\frac{1}{nx}} \cdot \frac{1}{x}\right) = \frac{1}{x}$ $\quad \boxed{f(x) = \frac{1}{x}}$ $\quad (x>0)$

d) $f_n(x) = \frac{nx}{1+n^2x^2}$ $\quad S = \mathbb{R}$

$x = 0 \Rightarrow$ $\quad f(0) = 0$

$x \neq 0$

$\lim_{n\to\infty} \frac{nx}{1+n^2x^2} = \lim_{n\to\infty} \frac{\frac{1}{n}x}{\frac{1}{n^2}+x^2} = \lim_{n\to\infty} \frac{\overset{\to 0}{\frac{x}{n}}}{\underset{0}{\frac{1}{n^2}}+x^2} = \frac{0}{x^2} = 0$

$f(x) = 0 \quad (\forall x \in \mathbb{R})$

e) $f_n(x) = \ln\left(3 + \frac{n^2e^x}{n^4+e^{2x}}\right)$ $\quad S = [0, +\infty)$

$\ln(1+t) \sim t \quad (t \to 0)$

$\lim_{n\to\infty} \left(\ln\left(3+\frac{n^2e^x}{n^4+e^{2x}}\right)\right) = \lim_{n\to\infty} \ln\left(3\cdot\left(1+\frac{n^2e^x}{3\cdot(n^4+e^{2x})}\right)\right)$

$= \lim_{n\to\infty} \left(\ln 3 + \ln\left(1+\frac{n^2e^x}{3\cdot(n^4+e^{2x})}\right)\right) =$

$= \lim_{n\to\infty} \left(\ln 3 + \ln\left(1+\underbrace{\frac{n^2e^x}{3n^4+3e^{2x}}}_{t}\right)\right) \sim$

$\sim \ln 3 + \frac{n^2 e^x}{3n^4 + 3e^{2x}} \sim \ln 3 + \frac{n^2 e^x}{3n^4} \sim \ln 3 + \frac{e^x}{3n^2} \to \ln 3$

Za vježbu:

f) $f_n(x) = x^n \cdot \cos \frac{1}{nx}$ $\quad S = [0, +\infty)$

g) $f_n(x) = \sqrt{x^2 + \frac{1}{\sqrt{n}}}$ $\quad S = \mathbb{R}$

h) $f_n(x) = n\left(x^{\frac{1}{n}} - 1\right)$ $\quad S = [1, 3]$

i) $f_n(x) = \sqrt[n]{1 + x^n}$ $\quad S = [0, 2]$

j) $f_n(x) = (x-1) \operatorname{arctg} x^n$, $\quad S = (0, +\infty)$

2 Dokazati da je funk. niz $f_n(x) = n x^n$ konvergentan na skupu $S_1 = [0, \frac{1}{2}]$ a divergentan na skupu $S_2 = [0, 1]$

$x = 0 \quad f_n(0) = n \cdot 0 = 0 \to \quad (n \to \infty)$

$f_n(1) = n \cdot 1^n = 1 \to \infty \quad (n \to \infty)$

$\Rightarrow f_n$ divergira na skupu S_2

$f_n(x) = n \cdot x^n$

Neka je $x_0 \in [0, \frac{1}{2}]$ proizvoljno

Ako red konvergira njegov opšti član $\to 0$

Za $x_0 = 0 \Rightarrow f_n(x_0) = 0$

Ako je $x_0 \neq 0$ dokazaćemo da $f_n(x_0) \to 0 \ (n \to \infty)$

$\lim\limits_{n\to\infty} n \cdot x_0^n = 0$.

Posmatrajmo pozitivni red $\sum\limits_{n=1}^{\infty} n \cdot x_0^n$

Pošto je $\sqrt[n]{n \cdot x^n} = \sqrt[n]{n} \cdot \sqrt[n]{x^n} = x_0 \cdot \underbrace{\sqrt[n]{n}}_{\to 1 \ (n\to\infty)} \to x_0 \leq \frac{1}{2} < 1$,

Prema Košijevom korjenom kriteriju dati red konvergira i zato je $\lim\limits_{n\to\infty} n \cdot x_0^n = 0$

$f_n \to 0$ na S (f_n konvergira ka nuli).

3. Ispitati ravnomjernu konvergenciju datog niza na datom skupu:

a) $f_n(x) = x^n - x^{n+1}$ $\qquad S = [0,1]$

$f(x) = \underbrace{x^n}_{\geq 0} \underbrace{(1-x)}_{\geq 0}$

$x^n = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases} \Rightarrow f_n(x) \to 0 \quad (n \to \infty)$

$f(x) = 0 \quad (x \in S)$

$\sup\limits_{x\in S} |\underbrace{f_n(x)}_{\geq 0} - \underbrace{f(x)}_{0}| = \sup |\underbrace{f_n(x)}_{\geq 0}| = \sup f_n(x) =$

$$f_n'(x) = n\cdot x^{n-1} - (n+1)\cdot \overset{= x^{n-1}\cdot x}{x^n} = \underbrace{x^{n-1}}_{\geq 0}\left(n-(n+1)\cdot x\right)$$

$$f_n'(x) = 0 \Rightarrow x = 0 \quad \vee \quad n = (n+1)x \qquad x = \frac{n}{n+1}$$

	0		$\frac{n}{n+1}$		1
x^{n-1}		$+$		$+$	
$n-(n+1)x$		$+$	0	$-$	
f_n'		$+$		$-$	
f_n		$\nearrow$	max	$\searrow$	

$\leftarrow x = 0 \Rightarrow +$

$x = 1 \Rightarrow -$

Ako je f-ja neprekidna na zat. intervalu onda će ona na tom int. dostići svoj supremum.

$$f\left(\frac{n}{n+1}\right) = \left(\frac{n}{n+1}\right)^n \cdot \left(1 - \frac{n}{n+1}\right) = \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1}$$

$$\Rightarrow \max_{x\in S} f_n(x) = \sup_{x\in S} f_n(x) = \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1} \qquad \Big| : \frac{n}{n}$$

$$\lim_{n\to\infty} \sup_{x\in S} f_n(x) = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^n \cdot \frac{1}{n+1} = e\cdot 0 = 0$$

$$f_n \underset{S}{\rightrightarrows} 0$$

b) $f_n(x) = \dfrac{n^2}{n^2+x^2}$ $\qquad S = [-1, 1]$

$f_n \underset{S}{\rightrightarrows} 1$ - dok. u I zad pod b)

$$\left| f_n(x) - f(x) \right| = \left| \frac{n^2}{n^2+x^2} - 1 \right| = \left| \frac{-x^2}{n^2+x^2} \right| = \frac{x^2}{n^2+x^2} \leq \frac{x^2}{n^2} \leq \frac{1}{n^2}$$

Ispunjen je uslov za ravn. kon. funkc. niza.

$$f_n \underset{S}{\rightrightarrows} 1$$

c) $f_n(x) = \frac{\sin nx}{n}$ $S = \mathbb{R}$

$$-\frac{1}{n} < \frac{\sin nx}{n} < \frac{1}{n}$$

$$0$$

$$f(x) = 0$$

$$\left| f_n(x) - \underbrace{f(x)}_{0} \right| = \left| \frac{\sin nx}{n} \right| \leq \frac{1}{n}$$ li

$$\sup_{x \in S} \left| f_n(x) - f(x) \right| = \frac{1}{n}$$

$\lim\limits_{n \to \infty} \sup\limits_{x \in S} | \ldots | = 0$ - f_n ravnomjerno konvergira na S

$$f \underset{S}{\rightrightarrows} 0$$

d) $f_n(x) = \sin \frac{x}{n}$, $x \in \mathbb{R}$

$$f(x) = 0 \quad (x \in \mathbb{R})$$

$$\lim_{n \to \infty} \sup_{x \in S} \left| f_n(x) - \underbrace{f(x)}_{0} \right| = \lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left| \sin \frac{x}{n} \right| = 1$$

$$f_n \underset{S}{\not\rightrightarrows} 0$$

x^2 na $(-1,1)$ je uvjet < 1

$\Rightarrow 0$

e) $f_n(x) = \frac{nx}{1+n^2x^2}$ $\quad S = [0,2]$

$f(x) = 0 \qquad f_n \underset{S}{\to} 0$

$x_n = \frac{1}{n} \quad (n \in \mathbb{N}) \Rightarrow x_n \in S$

$\left| f_n(x_n) - \underbrace{f(x_n)}_{0} \right| = \frac{n \cdot \frac{1}{n}}{1 + n^2 \cdot \frac{1}{n^2}} = \frac{1}{2}$ Nema ravn. konvergencije

jer $f(x) \to 0$

jer $f_n(x_n)$ mora $\to 0$ a kod nas je to uvijek $\frac{1}{2}$.

f) Posmatrajmo niz: $f_n(x) = \frac{x}{x+n}$ $\quad S_1 = [0, a]$, $a > 0$

$S_2 = [0, +\infty)$

$$0 \le |f_n(x)| \le \frac{x}{n} \to 0$$

$$0$$

$f_n(x) \underset{S_1, S_2}{\to} 0 \quad (n \to \infty)$

$|f_n(x) - \underbrace{f(x)}_{0}| \le \frac{x}{n} \le \frac{a}{n} \to 0$ ako $x \in S_1$

$f_n \underset{S_1}{\rightrightarrows} 0$ (na S_1)

$x_n = n \quad (n \in \mathbb{N})$

$\left| f_n(x_n) - \underbrace{f(x_n)}_{0} \right| = \frac{n}{n+n} = \frac{n}{2n} = \frac{1}{2}$

jer $f(x) = 0$

(ako ravnomjerno kon. ova razlika bi trebala $\to 0$ a ona je $\frac{1}{2}$ pa ne konv. ravnomjerno)

g) $f_n(x) = \frac{x^n}{1+x^n}$

$S_1 = (0, 1-\delta)$

$S_2 = (1+\delta, \infty)$

$S_3 = (0, 1)$

$S_4 = (1, +\infty)$ $\quad \delta > 0$

$S_5 = (1-\delta, \infty)$

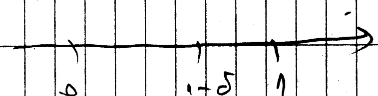
$x \in S_1 \Rightarrow x^n \to 0$

$\Rightarrow f(x) = 0$

$|f_n(x) - f(x)| = \left|\frac{x^n}{1+x^n} - 0\right| = \frac{x^n}{1+x^n} \le \frac{x^n}{1} = x^n \le (1-\delta)^n \to 0$

jer $1-\delta \in (0,1)$

$f_n \underset{S_1}{\rightrightarrows} 0$



$\lim\limits_{n\to\infty} \sup\limits_{x\in S_3} |f_n(x) - f(x)| = \frac{1}{2}$ tako $f_n \underset{S_3}{\not\rightrightarrows} 0$

$x \in S_2 \lor x \in S_4 \Rightarrow x > 1 \Rightarrow x^n \to \infty$

$\lim\limits_{n\to\infty} \frac{x^n}{1+x^n} = \frac{:x^n}{:x^n} = \frac{1}{\underbrace{\frac{1}{x^n}}_{0}+1} = 1$

f_n konvergira ka 1 na skupovima S_2 i S_4

za vježbu

$|f_n(x) - \underbrace{f(x)}_{1}| = \dots$

na S_2 ima ravn. kon, na S_4 nema

S_5 - $f(x)$ neće bit ni 0 ni 1

g_5 : $f(x) \begin{cases} 0, & x \in (1-\delta, 1) \\ 1, & x > 1 \end{cases}$

h) $f_n(x) = \dfrac{\operatorname{arctg} nx}{\sqrt{n+x}}$ $\quad S = [0, +\infty)$

$\dfrac{0}{\sqrt{n+x}} \le \dfrac{\operatorname{arctg} nx}{\sqrt{n+x}} \le \dfrac{\frac{\pi}{2}}{\sqrt{n+x}}$ $\quad f(x) = 0$

$\searrow \quad \downarrow \quad \swarrow$

0

$|f_n(x) - f(x)| = \left| \dfrac{\operatorname{arctg} nx}{\sqrt{n+x}} - 0 \right| = \dfrac{\operatorname{arctg} nx}{\sqrt{n+x}} \le \dfrac{\frac{\pi}{2}}{\sqrt{n}} \to 0$ $(n \to \infty)$

$f_n(x) \underset{S}{\rightrightarrows} 0$

za vježbu:

i) $f_n(x) = \dfrac{x + xn^3 + x^3 n^6}{1 + x^2 n^6}$ $\quad S \in [1, +\infty)$

j) $f_n(x) = x^n + x^{2n} - x^{3n}$ $\quad S = [0, 1]$

k) $f_n(x) = nx \cdot (1-x)^n$ $\quad S = [0, 1]$

l) $f_n(x) = \dfrac{nx^2}{1 + 2n + x}$ $\quad S_1 = [0, 1]$, $\quad S_2 = [1, +\infty]$